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Study of Numerical solution of Ordinary Differential Equation by Taylor, Euler and Runge-Kutta methods

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Abstract: **Numerical solution methods of ordinary differential equations in numerical analysis is an important topic, which is usually used for many differential equations that is difficult to find their exact and analytic solution or the equation which cannot be represented in explicit form. There are many methods of numerical solution of ordinary differential equations such as; Taylor method, Euler method, Hunn method and Runge-Kutta method with first, second, third, fourth and higher orders respectively. Taylor's method is very accurate for numerical solution of differential equations, but it is rarely used because of the need for computations of successive derivatives. Euler's method has more errors but needs less computation. The Runge-Kutta method is a suitable and the most commonly used method with less computational steps and accurate calculation. The Runge-Kutta method is the generalized form of the Euler method which is used for numerical solution of ordinary differential equations. In this paper, the numerical solutions of ordinary differential equations are solved by Taylor, Euler and Runge-Kutta fourth-order methods and then their exact solutions are compared using tables and graphs.**

Keywords: **Ordinary Differential Equations, Numerical Solution of Equations, Exact Solution of Equations, Taylor's Method, Euler's Method, Runge-Kutta Method.**

I. INTRODUCTION

Ordinary differential equations are one of the important and widely used techniques in mathematical modeling. However, not many ordinary differential equations have an analytic solution, usually it is extremely difficult to obtain and it is not very practical [1].

Differential equations are the best language for expressing many of the general laws of nature in quantum physics, electronics, computational chemistry and astronomy [2]. Therefore, solution of these equations is of particular importance. Numerical solution methods are also of particular importance in applied problems. Finding numerical solution of differential equations is an important topic in numerical analysis, which is usually used for many differential equations that is difficult to find their exact and analytic solution or the equation which cannot be represented in explicit form. This problem may because of the nonlinear equations or may they have coefficients that change over time. For example, the linear differential equations having as much as higher coefficients, the more difficult it is to solve. Some equations are also more difficult to solve because of more inputs under different conditions [3].

There are many methods like; Taylor, Euler, Heun, Multistep, Adams-Bashforth, Adams-mouton, Runge-Kutta methods which produce numerical approximations to solution of initial value problem in ordinary differential equation. Euler's method which is the oldest and simplest method originated by Leonhard Euler in 1768 and improved Euler method, Runge Kutta methods described by Carl Runge and Martin Kutta in 1895 and 1905, respectively [4]. Therefore, finding numerical solutions of differential equations by Taylor's method has proper accuracy but is rarely used because of the need for successive derivatives computation, but the fourth-order Runge-Kutta method is the most commonly used technique in numerical solution of differential equations.

II. LITERATURE REVIEW

For finding numerical solution of ordinary differential equation in general, each numerical method has its own advantages and disadvantages of use [4].

Taylor's method is one of the best methods and have proper accuracy but rarely used because of the need successive derivatives calculations [5].

Runge–Kutta methods have been presented for the integration of linear systems of ordinary differential equations with constant coefficients. when the step size is limited by stability, then the fourth-order method is the most suitable [6].

Runge-Kutta method and Usmani Agarwal method are compared with a new method for numerical solution of three problems and the result shows that in all three problems with step size of $h = 0.1$ and $h = 0.05$ the accuracy of new method is more than Usmani Agarwal and Runge Kutta methods but with step size of $h=0.2$ Usmani Agarwal method has more accuracy than the new method [7].

Numerical methods for systems of first order ordinary differential equations are tested on a variety of initial value problems. In this case Runge-Kutta methods are not competitive, but fourth or fifth order methods of this type are best for restricted classes of problems in which function evaluations accuracy requirements are not very stringent [8].

Numerical solution of linear and nonlinear equations are compared with Adomian decomposition and Runge-Kutta methods and the result shows that Adomian decomposition method is very powerful [9].

Adams-Moulton and Runge-Kutta-Merson Methods which are used for solving initial-value problems in ordinary differential equations are improved in case of efficiency by the Modified Taylor method based on three derivatives [10].

Taylor's method is accurate but it is less commonly used because of its successive derivatives computation. The Euler's method is also a suitable method, but the error is more in this method. The Runge-Kutta method has different order and is more accurate than other methods and has less error.

III. NUMERICAL SOLUTION METHODS

A. *Taylor's method*

For finding the answer of differential equation $y' = f(x, y)$ with initial-value $y(x_0) = y_0$ in closed interval $[a,b]$ we follow [11]:

I. Partition the interval $[a,b]$ into n equal parts with length $h = \dfrac{b-a}{n}$.

$x_0 = a$, $x_n = b$, $y(x_n) = y(a + nh)$, $x_n = a + nh$

II. With y_n we obtain numerical value $y(x_{n+1})$ i.e. y_{n+1} from following formula:

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!} f'(x_n, y_n) + \dots + \frac{h^p}{p!} f^{(p-1)}(x_n, y_n) \quad , \quad n = 0,1,2,\dots,n-1$$

e.g. find the numerical solution $y(1.5)$ of ordinary differential equation $y' = 2xy,\ y(1) = 1$ for $P = 4$ with step size $h = 0.1$ by Taylor's method.

Sol. Since $x_0 = 1, y_0 = 1$ and $f(x, y) = 2x \cdot y$ then we expand Taylor's series up to forth order:

$$y_{n+1} = y_n + hf(x_n, y_n) + \frac{h^2}{2!} f'(x_n, y_n) + \frac{h^3}{3!} f''(x_n, y_n) + \frac{h^4}{4!} f'''(x_n, y_n) \ , \ n = 0,1,2,3$$

By differentiating first up to forth order of function $f(x, y) = 2x \cdot y$ at the point $(x_0, y_0) = (1,1)$ we have:

$y_0' = 2$, $y_0'' = 6$, $y_0''' = 20$, $y_0^{(4)} = 76$

Now we put these values in Taylor's series, then for $n = 0$ we obtain value y_1:

$$y_1 = y_0 + h\, f(x_0, y_0) + \frac{h^2}{2!} f'(x_0, y_0) + \frac{h^3}{3!} f''(x_0, y_0) + \frac{h^4}{4!} f'''(x_0, y_0) = 1.2336496$$

i.e. $y(x_1) \simeq y_1 = 1.23365$.

By differentiating first up to forth order of function $f(x, y) = 2x \cdot y$ at the point $(x_1, y_1) = (1.1, 1.23365)$ we have:

$y_1' = 2.71403$, $y_1'' = 8.438166$, $y_1''' = 29.4200852$, $y_1^{(4)} = 115.35318344$

Now we want to find value of y_2 such that $x_1 = x_0 + h = 1.1$ then according to Taylor series for $n = 1$ we have:

$$y_2 = y_1 + h\, f(x_1, y_1) + \frac{h^2}{2!} f'(x_1, y_1) + \frac{h^3}{3!} f''(x_1, y_1) + \frac{h^4}{4!} f'''(x_1, y_1) = 1.55262783655$$

i.e. $y(x_2) \simeq y_2 = 1.55263$.

Similarly, we get the values of y_3, y_4, y_5 after calculation as followings:

$y_3 = 1.9936$, $y_4 = 2.6116$, $y_5 = 3.4902$

The Taylor's method for numerical solution of $y(1.5)$ in ordinary differential equation $y' = 2xy,\ y(1) = 1$ with step size $h = 0.1$ for $P = 4$ with exact solution and absolute error is shown in table 1.

TABLE I: Numerical solution of equation y' = 2xy, y(1) = 1 by Taylor's method with step size h=0.1

X_n	Exact Value	Y_n(Taylor)	Absolute Error
1.00	1.0000	1.0000	0.0000
1.10	1.2337	1.2336	0.0001
1.20	1.5527	1.5526	0.0001
1.30	1.9937	1.9936	0.0001
1.40	2.6117	2.6116	0.0001
1.50	3.4904	3.4902	0.0002

Graph of numerical and exact solution of above equation for $y(1.5)$ by Taylor's method is as following [2].

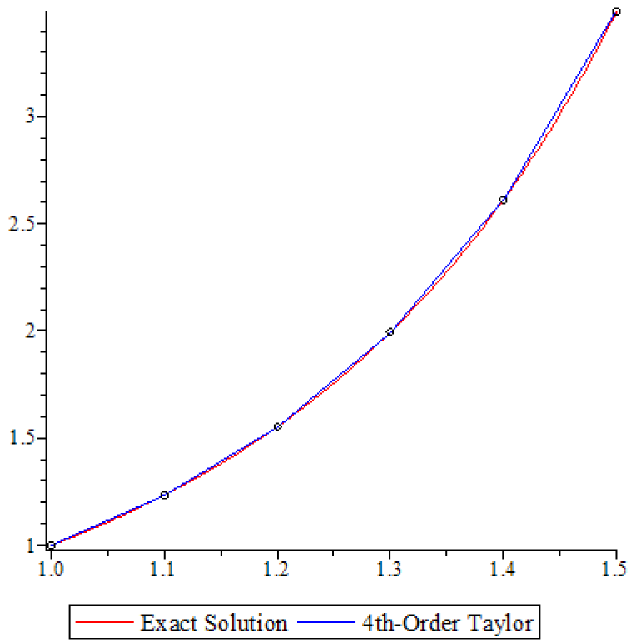

Figure I: Graph of Numerical and exact solution of equation y' = 2xy, y(1) = 1 for y(1.5) by Taylor's method.

B. Euler's method

If we put $p=1$ in Taylor's method, then Euler's method will be obtained which has the formula, $y_{n+1}=y_n+h\,f(x_n,y_n)$ such that f is the function which is obtained from equation $y'=f(x,y)$, h is a positive number or difference between x_n and x_{n+1}. Values of $y_1,y_2,...,y_n$ are numerical solution of $y(x)$ at $x_1,x_2,...,x_n$.

Therefore, Euler's method is the same as Taylor's method [12].

e.g. find the numerical solution $y(1.5)$ of ordinary differential equation $y'=2xy,\ y(1)=1$ with step size $h=0.1$ by Euler's method.

Sol. Since $x_0=1, y_0=1$ and $f(x,y)=2x\cdot y$, then by Euler's formula we have, $y_{n+1}=y_n+h\,f(x_n,y_n)$ for $n=0,1,2,3,4$ we calculate the values of y_1,y_2,y_3,y_4,y_5, then:

$y_1=1.2$, $y_2=1.464$, $y_3=1.81536$, $y_4=2.28735$, $y_5=2.92781$

The numerical solution of $y(1.5)$ in ordinary differential equation $y'=2xy,\ y(1)=1$ with step size $h=0.1$ by Euler's method with exact solution and absolute error is shown in table 2.

TABLE II: Numerical solution of equation y' = 2xy, y(1) = 1 by Euler's method with step size h=0.1

X_n	Exact Value	Y_n(Euler)	Absolute Error
1.00	1.0000	1.0000	0.0000
1.10	1.2337	1.2000	0.0337
1.20	1.5527	1.4640	0.0887
1.30	1.9937	1.8154	0.1783
1.40	2.6117	2.2874	0.3243
1.50	3.4904	2.9278	0.5626

From table 1, We can observe that absolute errors by Euler's method are more than Taylor's method. To reduce the errors, h should be considered small. But Euler's method has less calculation complexity.

The Following graph depicts the numerical and exact solution of above differential equation for $y(1.5)$ by Euler's method [2].

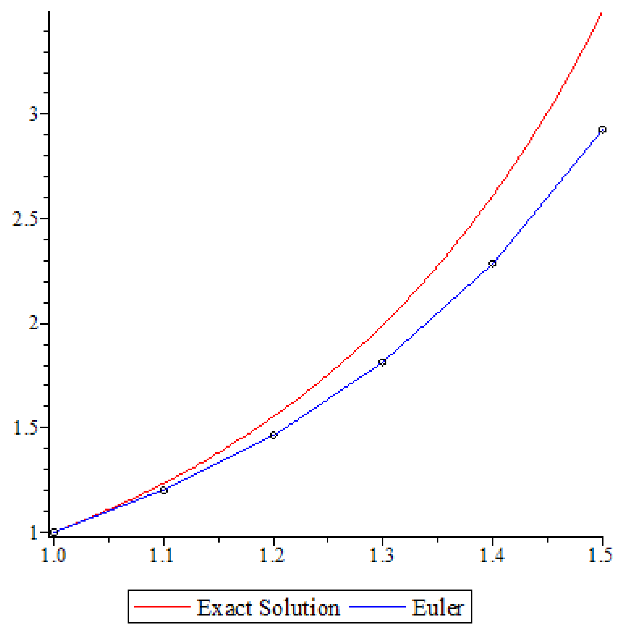

Figure II: Graph of Numerical and exact solution of equation y' = 2xy, y(1) = 1 for y(1.5) by Euler's method.

C. Fourth Order Runge-Kutta Method

We start from ordinary differential equation $y' = f(x, y)$ with initial-value $y(x_0) = y_0$, the following formula is the Euler's method for numerical solution of ordinary differential equation [13].

$$y_{n+1} = y_n + hf(x_n, y_n)......................(1)$$

Such that f is the function which is obtained from equation $y' = f(x, y)$. h is a positive number or difference between x_n and x_{n+1}. Values of $y_1, y_2, ..., y_n$ are numerical solution of $y(x)$ at $x_1, x_2, ..., x_n$ [14].

Basically, all Runge-Kutta methods are generalizations of the following basic Euler formula [7] [15]:

$$y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + ... + w_mk_m)......................(2)$$

Values of $w_1, w_2, ..., w_m$, $i = 1, 2, 3, ..., m$ are constant that generally satisfy $w_1 + w_2 + ... + w_m = 1$ and $k_i (i = 1, 2, 3, ..., m)$ the function f evaluated at a selected point (x, y). The number m is called the order of the method.

Suppose if $m = 1, w_1 = 1$ and $k_1 = f(x_n, y_n)$, then we get the familiar Euler formula $y_{n+1} = y_n + h\, f(x_n, y_n)$. Hence Euler's method is said to be a first order Runge-Kutta method [16].

If we select $m = 1, 2, 3, 4$ in (2), then formula (3) with constants $w_1, w_2, w_3, w_4, \alpha_1, \alpha_2, \alpha_3$ and $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5, \beta_6$ are called fourth order Runge-Kutta method.

$$y_{n+1} = y_n + h(w_1k_1 + w_2k_2 + w_3k_3 + w_4k_4)............(3)$$

Where,

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \alpha_1 h, y_n + \beta_1 hk_1)$$

$$k_3 = f(x_n + \alpha_2 h, y_n + \beta_2 hk_1 + \beta_3 hk_2)$$

$$k_4 = f(x_n + \alpha_3 h, y_n + \beta_4 hk_1 + \beta_5 hk_2 + \beta_6 hk_3)$$

In accordance with fourth degree polynomial of Taylor's series of equation system from above parameters eleven equations and thirteen unknowns will be formed which has infinite solutions. We can get the values of parameters after solution of equation system as follows [16]:

$$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

$$k_1 = f(x_n, y_n)$$

$$k_2 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1)$$

$$k_3 = f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_2)$$

$$k_4 = f(x_n + h, y_n + hk_3)$$

e.g. find the numerical solution $y(1.5)$ of ordinary differential equation $y' = 2xy,\ y(1) = 1$ with step size $h = 0.1$ by fourth order Runge-Kutta method.

Sol. For the sake of illustration let us compute the case when $n = 0$, from above formula we have:

$k_1 = 2$, $k_2 = 2.31$, $k_3 = 2.34255$, $k_4 = 2.715361$

Then $y_1 = y_0 + \frac{0.1}{6}(k_1 + 2k_2 + 2k_3 + k_4) = 1 + \frac{0.1}{6}[2 + 2(2.31) + 2(2.34255) + 2.715361] = 1.23367435$

i.e. $y(x_1) \simeq y_1 = 1.23367$.

Similarly, values of y_2, y_3, y_4, y_5 with exact solution and absolute error is shown in table 3 [17].

TABLE III: Numerical solution of equation y'=2xy, y(1)=1 by 4th order Runge-Kutta method with step size h=0.1

X_n	Exact Value	Y_n(RK4)	Absolute Error
1.00	1.0000	1.0000	0.0000
1.10	1.2337	1.2337	0.0000
1.20	1.5527	1.5527	0.0000
1.30	1.9937	1.9937	0.0000
1.40	2.6117	2.6116	0.0001
1.50	3.4904	3.4902	0.0002

It can be observed in Table 3 that absolute error of the numerical solution of equation by fourth-order Rang-Kutta method is zero or close to zero [16].

Following graph is the numerical and exact solution of above differential equation for $y(1.5)$ by fourth order Runge-Kutta method [2].

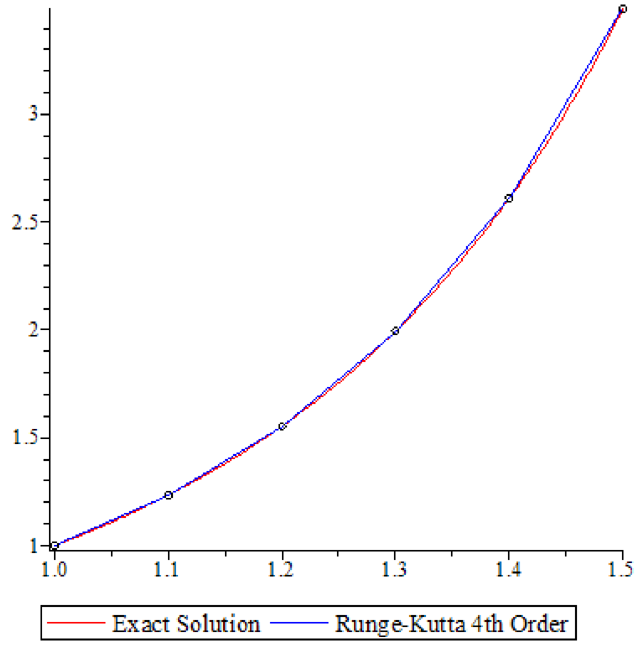

Figure III: Graph of Numerical and exact solution of equation y' = 2xy, y(1) = 1 for y(1.5) by fourth order Runge-Kutta method.

IV. COMPARISON OF NUMERICAL SOLUTION METHODS

A. Comparison of Numerical Solutions of Differential Equation $y' = 2xy,\ y(1) = 1$ by Taylor, Euler and Fourth-Order Rang-Kutta Method

The main criteria for comparing methods are the accuracy of the answers and the Volume rate of computation [18].

Table 4 shows the comparison of Numerical and exact solution of ordinary differential equation $y' = 2xy,\ y(1) = 1$ with step size $h = 0.1$ according to absolute errors by the above three methods.

TABLE IV: Numerical solution of equation y'=2xy, y(1)=1 by Taylor, Euler and fourth order Runge-Kutta method with step size h=0.1

X_n	Exact Value	Y_n(Taylor)	Error	Y_n(Euler)	Error	Y_n(RK4)	Error
1.00	1.0000	1.0000	0.0000	1.0000	0.0000	1.0000	0.0000
1.10	1.2337	1.2336	0.0001	1.2000	0.0337	1.2337	0.0000
1.20	1.5527	1.5526	0.0001	1.4640	0.0887	1.5527	0.0000
1.30	1.9937	1.9936	0.0001	1.8154	0.1783	1.9937	0.0000
1.40	2.6117	2.6116	0.0001	2.2874	0.3243	2.6116	0.0001
1.50	3.4904	3.4902	0.0002	2.9278	0.5626	3.4902	0.0002

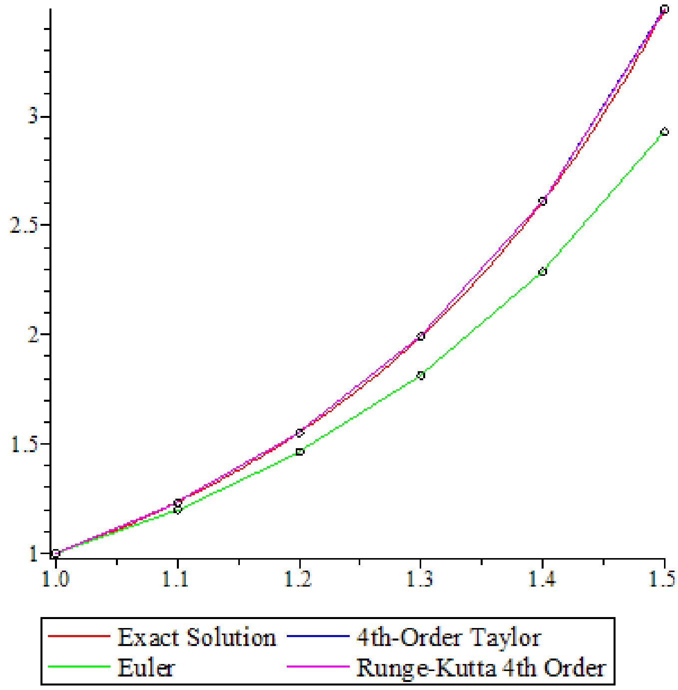

Figure IV: Graph of Numerical and exact solution of equation y' = 2xy, y(1) = 1 for y(1.5) by Taylor, Euler and fourth order Runge-Kutta method.

V. CONCLUSION

In this paper Taylor, Euler and Fourth-Order Rang-Kutta Methods for numerical solution of ordinary differential equations and the comparison of numerical solutions of differential equation $y' = 2xy,\ y(1) = 1$ by mentioned methods are discussed and the result shows that Taylor's method has more accuracy but it requires long calculations, Euler's method is suitable method with less calculation but absolute errors in Euler's method is more than other two methods and this is one of the disadvantages of Euler's method.

To reduce the errors, the value of h should be chosen small. Comparison of numerical and exact solutions shows that the solution of differential equation by fourth-order Runge-Kutta method is very close to the exact equation and has less error. Change of value h have effect on the numerical solution of differential equation by fourth order Runge-Kutta method, it means if the value of h is selected small then numerical solution errors according exact errors will be less. The problem with choosing a small value for h by fourth order Runge-Kutta method is that, it has more calculation and takes more time. The advantage of choosing a small value for h is that the obtained solution has high accuracy and close to the general solution. If the value of h is chosen large then the long calculation is prevented.

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