A New Approach to Study Fractional Integral Problems

Chii-Huei Yu

School of Mathematics and Statistics, Zhaoqing University, Guangdong Province, China

Abstract: **Fractional calculus is widely used in many scientific fields. This paper uses a new method to study fractional integral calculus. Based on the Jumarie type of modified Riemann-Liouville (R-L) fractional derivative, we make use of a new multiplication and some techniques include integration by parts for fractional calculus to solve some problems of fractional integral. The modified Jumarie's R-L fractional derivative is closely related to classical calculus, which can make the fractional derivative of constant function to zero. Seven kinds of special fractional integral problems are provided, and the results we obtained are the generalizations of classical integral problems. Furthermore, Mittag-Leffler function plays an important role in this study, which is similar to the exponential function in traditional calculus. On the other hand, several examples are given to illustrate our results.**

Keywords: **Jumarie type of modified R-L fractional derivative, New multiplication, Integration by parts for fractional calculus, Problems of fractional integral, Mittag-Leffler function.**

I. INTRODUCTION

In recent decades, the applications of fractional calculus in various fields of science is growing rapidly, such as applied mathematics, physics, mathematical biology, mechanics, engineering, elasticity, dynamics, control theory, electronics, modelling, probability, finance, economics, chemistry, etc [1-14]. But the definition of fractional derivative is not unique, many authors have given the definition of fractional derivative. The common definition is Riemann-Liouville (R-L) fractional derivative [15]. Other useful definitions include Caputo fractional derivative [16], Grunwald-Letnikov fractional derivative [17], Jumarie's modification of R-L fractional derivative [18-19]. Riemann-Liouville definition the fractional derivative of a constant is non-zero which creates a difficulty to relate between the classical calculus. To overcome this difficulty, Jumarie [19] modified the definition of fractional derivative of Riemann-Liouville type and with this new formula, we obtain the derivative of a constant as zero. Thus, it is easier to connect fractional calculus with classical calculus by using this definition. On the other hand, by using the Jumarie modified definition of fractional derivative, it is obtained that the derivative of Mittag-Leffler function is Mittag-Leffler function itself. Like the classical derivative, the derivative of exponential function is exponential function itself. Therefore, the modified R-L fractional derivative of Jumarie type has a conjugate relationship with classical calculus. In many cases, it is easy to solve the fractional differential equations based on Jumarie fractional derivative [23-29].

In this paper, based on the Jumarie's modified R-L fractional derivatives, a new fractional function multiplication is defined, and several fractional integral problems are studied by using some methods include integration by parts for fractional calculus. In fact, these results are the generalizations of classical calculus. In addition, some examples are provided to demonstrate the advantage of our results. This article studies the following seven types of fractional integrals:

$$\left({}_0I_x^{\alpha}\right)\left[\left(c\cdot sin_{\alpha}(x^{\alpha})+d\cdot cos_{\alpha}(x^{\alpha})\right)\otimes\left(a\cdot sin_{\alpha}(x^{\alpha})+b\cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes-1}\right], \tag{1}$$

$$\left({}_0I_b^{\alpha}\right)\left[f(x^{\alpha})\otimes\left(f(x^{\alpha})+f(b-x^{\alpha})\right)^{\otimes-1}\right], \tag{2}$$

$$\left({}_0I_x^{\alpha}\right)\left[\left(c+d\cdot E_{\alpha}(x^{\alpha})\right)\otimes\left(a+b\cdot E_{\alpha}(x^{\alpha})\right)^{\otimes-1}\right], \tag{3}$$

$$\left({}_0I^{\alpha}_{\frac{T_\alpha}{2}}\right)\left[\frac{1}{\Gamma(\alpha+1)}x^\alpha \otimes f\left(sin_\alpha(x^\alpha)\right)\right], \tag{4}$$

$$\left({}_0I^{\alpha}_{x}\right)\left[\left(b\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}+c\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha+d\right)\otimes\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes 2}\right)^{\otimes -1}\right], \tag{5}$$

$$\left({}_0I^{\alpha}_{+\infty}\right)[E_\alpha(-ax^\alpha)\otimes cos_\alpha(bx^\alpha)], \tag{6}$$

$$\left({}_0I^{\alpha}_{+\infty}\right)[E_\alpha(-ax^\alpha)\otimes sin_\alpha(bx^\alpha)], \tag{7}$$

where a, b, c, d, α are real numbers, $0 < \alpha \le 1$, and $f(x^\alpha)$ is a continuous function.

II. PRELIMINARIES

First, we introduce the definition of fractional derivative used in this paper.

Definition 2.1: Let α be a real number and m, n be positive integers. The Jumarie type modified R-L fractional derivatives ([19]) is defined by

$$ {}_aD^{\alpha}_{x}[f(x)] = \begin{cases} \frac{1}{\Gamma(-\alpha)}\int_a^x (x-\tau)^{-\alpha-1}f(\tau)d\tau, & \text{if } \alpha < 0 \\ \frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_a^x (x-\tau)^{-\alpha}[f(\tau)-f(a)]\,d\tau & \text{if } 0 \le \alpha < 1 \\ \frac{d^m}{dx^m}\left({}_aD^{\alpha-m}_{x}\right)[f(x)], & \text{if } m \le \alpha < m+1 \end{cases} \tag{8}$$

where $\Gamma(u) = \int_0^\infty t^{u-1}e^{-t}dt$ is the gamma function defined on $u > 0$. Moreover, we define the fractional integral of (x), $\left({}_aI^{\alpha}_{x}\right)[f(x)] = \left({}_aD^{-\alpha}_{x}\right)[f(x)]$, where $\alpha > 0$ and $f(x)$ is called α-fractional integrable function.

Proposition 2.2 ([20])**:** *Assume that α, β, c are real numbers and $\beta \ge \alpha > 0$, then*

$$ {}_0D^{\alpha}_{x}\left[x^\beta\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}x^{\beta-\alpha}, \tag{9}$$

and

$$ {}_0D^{\alpha}_{x}[c] = 0. \tag{10}$$

Definition 2.3 ([21])**:** The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma(k\alpha+1)}, \tag{11}$$

where α is a real number, $\alpha > 0$, and z is a complex variable.

Definition 2.4 ([20])**:** Suppose that $0 < \alpha \le 1$ and x is a real variable. Then $E_\alpha(x^\alpha)$ is called the α- fractional exponential function, and the period of $E_\alpha(ix^\alpha)$ is denoted as T_α. Moreover, the α-fractional cosine and sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{2k\alpha}}{\Gamma(2k\alpha+1)}, \tag{12}$$

and

$$sin_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k x^{(2k+1)\alpha}}{\Gamma((2k+1)\alpha+1)}. \tag{13}$$

On the other hand, we define the α-fractional logarithm function as

$$Ln_\alpha(x^\alpha) = \left({}_1I^{\alpha}_{x}\right)\left[\left(\frac{1}{\Gamma(\alpha+1)}x^\alpha\right)^{\otimes -1}\right]. \tag{14}$$

And the α-fractional arctan function is

$$arctan_\alpha(x^\alpha) = \sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\cdot\frac{1}{\Gamma((2k+1)\alpha+1)}x^{(2k+1)\alpha}, \tag{15}$$

the α-fractional arccot function is

$$arccot_{\alpha}(x^{\alpha}) = \frac{T_{\alpha}}{4} - arctan_{\alpha}(x^{\alpha}), \tag{16}$$

where $|x^{\alpha}| < 1$.

Next, we introduce a new multiplication of fractional functions.

Definition 2.5 ([22])**:** Let z be a complex number, $0 < \alpha \leq 1$, j, l, k be non-negative integers, and a_k, b_k be real numbers, $p_k(z) = \frac{1}{\Gamma(k\alpha+1)} z^k$ for all k. The $\otimes$ multiplication is defined by

$$p_j(x^{\alpha}) \otimes p_l(y^{\alpha}) = \frac{1}{\Gamma(j\alpha+1)}(x^{\alpha})^j \otimes \frac{1}{\Gamma(l\alpha+1)}(y^{\alpha})^l = \frac{1}{\Gamma((j+l)\alpha+1)}\binom{j+l}{j}(x^{\alpha})^j (y^{\alpha})^l, \tag{17}$$

where $\binom{j+l}{j} = \frac{(j+l)!}{j!\,l!}$.

If $f(x^{\alpha})$ and $g(y^{\alpha})$ are two fractional functions,

$$f(x^{\alpha}) = \sum_{k=0}^{\infty} a_k\, p_k(x^{\alpha}) = \sum_{k=0}^{\infty} \frac{a_k}{\Gamma(k\alpha+1)}(x^{\alpha})^k, \tag{18}$$

$$g(y^{\alpha}) = \sum_{k=0}^{\infty} b_k\, p_k(y^{\alpha}) = \sum_{k=0}^{\infty} \frac{b_k}{\Gamma(k\alpha+1)}(y^{\alpha})^k, \tag{19}$$

then we define

$$f(x^{\alpha}) \otimes g(y^{\alpha}) = \sum_{k=0}^{\infty} a_k\, p_k(x^{\alpha}) \otimes \sum_{k=0}^{\infty} b_k\, p_k(y^{\alpha})$$
$$= \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} a_{k-m} b_m p_{k-m}(x^{\alpha}) \otimes p_m(y^{\alpha})\right). \tag{20}$$

Proposition 2.6: $f(x^{\alpha}) \otimes g(y^{\alpha}) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)} \sum_{m=0}^{k} \binom{k}{m} a_{k-m} b_m\, (x^{\alpha})^{k-m}(y^{\alpha})^m$. (21)

Definition 2.7: Let $\left(f(x^{\alpha})\right)^{\otimes n} = f(x^{\alpha}) \otimes \cdots \otimes f(x^{\alpha})$ be the n times product of the fractional function $f(x^{\alpha})$. If $f(x^{\alpha}) \otimes g(x^{\alpha}) = 1$, then $g(x^{\alpha})$ is called the $\otimes$ reciprocal of $f(x^{\alpha})$, and denoted as $\left(f(x^{\alpha})\right)^{\otimes -1}$.

Definition 2.8: If $f(z) = \sum_{k=0}^{\infty} a_k z^k$, $g(x^{\alpha}) = \sum_{k=0}^{\infty} b_k\, p_k(x^{\alpha})$, then

$$f_{\otimes}\left(g(x^{\alpha})\right) = \sum_{k=0}^{\infty} a_k \left(g(x^{\alpha})\right)^{\otimes k}. \tag{22}$$

Theorem 2.9 (integration by parts for fractional calculus): *Let* $0 < \alpha \leq 1$, a, b *be real numbers, then*

$$\left({}_{a}I_{b}^{\alpha}\right)\left[f(x) \otimes \left({}_{a}D_{x}^{\alpha}\right)[g(x)]\right] = f(x) \otimes g(x)|_{a}^{b} - \left({}_{a}I_{b}^{\alpha}\right)\left[g(x) \otimes \left({}_{a}D_{x}^{\alpha}\right)[f(x)]\right]. \tag{23}$$

III. MAIN RESULTS

In the following, we discuss seven kinds of fractional integral problems.

Theorem 3.1 *Suppose that* a, b, c, d, α *are real numbers,* $a^2 + b^2 \neq 0$, *and* $0 < \alpha \leq 1$. *Then*

$$\left({}_{x}I_{0}^{\alpha}\right)\left[\left(c \cdot sin_{\alpha}(x^{\alpha}) + d \cdot cos_{\alpha}(x^{\alpha})\right) \otimes \left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]$$
$$= \frac{ad-bc}{a^2+b^2} \cdot Ln_{\alpha}(|a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})|) + \frac{ad+bc}{a^2+b^2} \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha}, \tag{24}$$

Proof $\left({}_{x}I_{0}^{\alpha}\right)\left[\left(c \cdot sin_{\alpha}(x^{\alpha}) + d \cdot cos_{\alpha}(x^{\alpha})\right) \otimes \left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]$

$$= \left({}_{x}I_{0}^{\alpha}\right)\left[\left(A\left(a \cdot cos_{\alpha}(x^{\alpha}) - b \cdot sin_{\alpha}(x^{\alpha})\right) + B\left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right)\right) \otimes \left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]$$

$$= \left({}_{x}I_{0}^{\alpha}\right)\left[A\left(a \cdot cos_{\alpha}(x^{\alpha}) - b \cdot sin_{\alpha}(x^{\alpha})\right) \otimes \left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]$$

$$+\left({}_{x}I_{0}^{\alpha}\right)\left[B\left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right) \otimes \left(a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]$$

$$= A \cdot Ln_{\alpha}(|a \cdot sin_{\alpha}(x^{\alpha}) + b \cdot cos_{\alpha}(x^{\alpha})|) + B \cdot \frac{1}{\Gamma(\alpha+1)} x^{\alpha},$$

where $A=\frac{ad-bc}{a^2+b^2}$ and $B=\frac{ad+bc}{a^2+b^2}$ q.e.d.

Theorem 3.2 *If b,α are real numbers, $0<\alpha\le 1$, and $f(x^\alpha)$ is a continuous function. Then*

$$\left({}_0I_b^\alpha\right)\left[f(x^\alpha)\otimes\left(f(x^\alpha)+f(b-x^\alpha)\right)^{\otimes -1}\right]=\frac{b}{2\Gamma(\alpha+1)}. \tag{25}$$

Proof Let $t^\alpha=b-x^\alpha$, then

$$\left({}_0I_b^\alpha\right)\left[f(x^\alpha)\otimes\left(f(x^\alpha)+f(b-x^\alpha)\right)^{\otimes -1}\right]=\left({}_0I_b^\alpha\right)\left[f(b-t^\alpha)\otimes\left(f(t^\alpha)+f(b-t^\alpha)\right)^{\otimes -1}\right]. \tag{26}$$

Since $\left({}_0I_b^\alpha\right)[1]=\frac{b}{\Gamma(\alpha+1)}$, and

$$\left({}_0I_b^\alpha\right)\left[f(x^\alpha)\otimes\left(f(x^\alpha)+f(b-x^\alpha)\right)^{\otimes -1}\right]+\left({}_0I_b^\alpha\right)\left[f(b-t^\alpha)\otimes\left(f(t^\alpha)+f(b-t^\alpha)\right)^{\otimes -1}\right]=\left({}_0I_b^\alpha\right)[1]. \tag{27}$$

It follows that the desired result holds. q.e.d.

Theorem 3.3 *Let a,b,c,d,α be real numbers, $ab\neq 0$, and $0<\alpha\le 1$.*

$$\left({}_0I_x^\alpha\right)\left[\left(c+d\cdot E_\alpha(x^\alpha)\right)\otimes\left(a+b\cdot E_\alpha(x^\alpha)\right)^{\otimes -1}\right]=A\cdot Ln_\alpha(|a+b\cdot E_\alpha(x^\alpha)|)+B\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha, \tag{28}$$

where $A=\frac{ad-bc}{ab}$ *and* $B=\frac{c}{a}$.

Proof $\left({}_0I_x^\alpha\right)\left[\left(c+d\cdot E_\alpha(x^\alpha)\right)\otimes\left(a+b\cdot E_\alpha(x^\alpha)\right)^{\otimes -1}\right]$

$$=\left({}_0I_x^\alpha\right)\left[\left(c+d\cdot E_\alpha(x^\alpha)\right)\otimes\left(a+b\cdot E_\alpha(x^\alpha)\right)^{\otimes -1}\right]$$

$$=\left({}_xI_0^\alpha\right)\left[\left(A\left(b\cdot E_\alpha(x^\alpha)\right)+B\left(a+b\cdot E_\alpha(x^\alpha)\right)\right)\otimes\left(a+b\cdot E_\alpha(x^\alpha)\right)^{\otimes -1}\right]$$

$$=\left({}_xI_0^\alpha\right)\left[A\left(b\cdot E_\alpha(x^\alpha)\right)\otimes\left(a+b\cdot E_\alpha(x^\alpha)\right)^{\otimes -1}\right]+\left({}_xI_0^\alpha\right)\left[B\left(a+b\cdot E_\alpha(x^\alpha)\right)\otimes\left(a+b\cdot E_\alpha(x^\alpha)\right)^{\otimes -1}\right]$$

$$=A\cdot Ln_\alpha(|a+b\cdot E_\alpha(x^\alpha)|)+B\cdot\frac{1}{\Gamma(\alpha+1)}x^\alpha.$$ q.e.d.

Theorem 3.4 *If $0<\alpha\le 1$, T_α is the period of $E_\alpha(ix^\alpha)$, and $f(x^\alpha)$ is a continuous function. Then*

$$\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[x^\alpha\otimes f\left(sin_\alpha(x^\alpha)\right)\right]=\frac{T_\alpha}{4}\cdot\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[f\left(sin_\alpha(x^\alpha)\right)\right]. \tag{29}$$

Proof Let $t^\alpha=\frac{T_\alpha}{2}-x^\alpha$, then

$$\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[x^\alpha\otimes f\left(sin_\alpha(x^\alpha)\right)\right]$$

$$=-\left({}_{\frac{T_\alpha}{2}}I_0^\alpha\right)\left[\left(\frac{T_\alpha}{2}-t^\alpha\right)\otimes f\left(sin_\alpha\left(\frac{T_\alpha}{2}-t^\alpha\right)\right)\right]$$

$$=\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[\left(\frac{T_\alpha}{2}-t^\alpha\right)\otimes f\left(sin_\alpha(t^\alpha)\right)\right]$$

$$=\frac{T_\alpha}{2}\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[f\left(sin_\alpha(t^\alpha)\right)\right]-\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[t^\alpha\otimes f\left(sin_\alpha(t^\alpha)\right)\right]$$

$$=\frac{T_\alpha}{2}\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[f\left(sin_\alpha(x^\alpha)\right)\right]-\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[x^\alpha\otimes f\left(sin_\alpha(x^\alpha)\right)\right]. \tag{30}$$

Therefore,

$$\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[x^\alpha\otimes f\left(sin_\alpha(x^\alpha)\right)\right]=\frac{T_\alpha}{4}\cdot\left({}_0I_{\frac{T_\alpha}{2}}^\alpha\right)\left[f\left(sin_\alpha(x^\alpha)\right)\right].$$ q.e.d.

Theorem 3.5 *Assume that a, b, c, d, α be real numbers, $a \neq 0$, and $0 < \alpha \leq 1$.*

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(b\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+c\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+d\right)\otimes\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$=b\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{c}{2}\cdot Ln_{\alpha}\left(\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\right)+\frac{d-a^2b}{a}\cdot arctan\left(\frac{1}{a}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right). \quad (31)$$

Proof Since $\left(b\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+c\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+d\right)\otimes\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}$

$$=b+\frac{c}{2}\cdot 2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}+(d-a^2b)\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}. \quad (32)$$

It follows that

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(b\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}+c\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+d\right)\otimes\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$=\left({}_{0}I_{x}^{\alpha}\right)\left[b+\frac{c}{2}\cdot 2\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)\otimes\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}+(d-a^2b)\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$=b\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+\frac{c}{2}\cdot Ln_{\alpha}\left(\left(a^2+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\right)+\frac{d-a^2b}{a}\cdot arctan\left(\frac{1}{a}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right).$$

q.e.d.

Theorem 3.6 *If a, b, α be real numbers, $a > 0$, and $0 < \alpha \leq 1$. Then*

$$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})]=\frac{a}{a^2+b^2}, \quad (33)$$

and

$$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes sin_{\alpha}(bx^{\alpha})]=\frac{b}{a^2+b^2}. \quad (34)$$

Proof By integration by parts for fractional calculus, we have

$$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})]$$

$$=\frac{1}{b}\left(E_{\alpha}(-ax^{\alpha})\otimes sin_{\alpha}(bx^{\alpha})|_{0}^{+\infty}\right)+\frac{a}{b}\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes sin_{\alpha}(bx^{\alpha})]$$

$$=\frac{a}{b}\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes sin_{\alpha}(bx^{\alpha})] \quad (35)$$

$$=-\frac{a}{b^2}\left(E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})|_{0}^{+\infty}\right)-\frac{a^2}{b^2}\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})]$$

$$=\frac{a}{b^2}-\frac{a^2}{b^2}\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})]. \quad (36)$$

It follows that

$$\left(1+\frac{a^2}{b^2}\right)\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})]=\frac{a}{b^2}. \quad (37)$$

Therefore,

$$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes cos_{\alpha}(bx^{\alpha})]=\frac{a}{a^2+b^2}.$$

Taking this result into Eq. (35), we get

$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-ax^{\alpha})\otimes sin_{\alpha}(bx^{\alpha})]=\frac{b}{a^2+b^2}$. q.e.d.

IV. EXAMPLES

Example 4.1 Using Theorem 3.1 yields

$$\left({}_{x}I_{0}^{\alpha}\right)\left[\left(2\cdot sin_{\alpha}(x^{\alpha})+5\cdot cos_{\alpha}(x^{\alpha})\right)\otimes\left(3\cdot sin_{\alpha}(x^{\alpha})-4\cdot cos_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]$$

$$=\frac{23}{25}\cdot Ln_{\alpha}(|3\cdot sin_{\alpha}(x^{\alpha})-4\cdot cos_{\alpha}(x^{\alpha})|)+\frac{7}{25}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \quad (38)$$

Example 4.2 It follows from Theorem 3.2 that

$$\left({}_{0}I_{\frac{T_{\alpha}}{4}}^{\alpha}\right)\left[\left(sin_{\alpha}(x^{\alpha})\right)^{\otimes n}\otimes\left(\left(sin_{\alpha}(x^{\alpha})\right)^{\otimes n}+\left(cos_{\alpha}(x^{\alpha})\right)^{\otimes n}\right)^{\otimes -1}\right]=\frac{T_{\alpha}}{8\Gamma(\alpha+1)} \quad (39)$$

for any positive integer n.

Example 4.3 By Theorem 3.3, we have

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(6+5\cdot E_{\alpha}(x^{\alpha})\right)\otimes\left(3+7\cdot E_{\alpha}(x^{\alpha})\right)^{\otimes -1}\right]=-\frac{9}{7}\cdot Ln_{\alpha}(|3+7\cdot E_{\alpha}(x^{\alpha})|)+2\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}. \quad (40)$$

Example 4.4 We have the following result from Theorem 3.4,

$$\left({}_{0}I_{\frac{T_{\alpha}}{2}}^{\alpha}\right)\left[x^{\alpha}\otimes sin_{\alpha}(x^{\alpha})\otimes\left(1+\left(cos_{\alpha}(x^{\alpha})\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$=\frac{T_{\alpha}}{4}\cdot\left({}_{0}I_{\frac{T_{\alpha}}{2}}^{\alpha}\right)\left[sin_{\alpha}(x^{\alpha})\otimes\left(1+\left(cos_{\alpha}(x^{\alpha})\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$=-\frac{T_{\alpha}}{4}\cdot arctan\left(cos_{\alpha}(x^{\alpha})\right)\Big|_{0}^{\frac{T_{\alpha}}{2}}$$

$$=-\frac{T_{\alpha}}{4}\cdot\left(-\frac{T_{\alpha}}{8}-\frac{T_{\alpha}}{8}\right)$$

$$=\frac{{T_{\alpha}}^{2}}{16}. \quad (41)$$

Example 4.5 Using Theorem 3.5, we obtain

$$\left({}_{0}I_{x}^{\alpha}\right)\left[\left(3\cdot\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}-4\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}+5\right)\otimes\left(4+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)^{\otimes -1}\right]$$

$$=3\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}-2\cdot Ln_{\alpha}\left(\left(4+\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes 2}\right)\right)-\frac{7}{2}\cdot arctan\left(\frac{1}{2}\cdot\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right). \quad (42)$$

Example 4.6 From Theorem 3.6, we get that

$$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-2x^{\alpha})\otimes cos_{\alpha}(3x^{\alpha})]=\frac{2}{13}. \quad (43)$$

And

$$\left({}_{0}I_{+\infty}^{\alpha}\right)[E_{\alpha}(-4x^{\alpha})\otimes sin_{\alpha}(7x^{\alpha})]=\frac{7}{65}. \quad (44)$$

V. CONCLUSION

In this present paper, a new multiplication and several techniques that include integration by parts for fractional calculus is used to evaluate some fractional integrals. In fact, the application of integration by parts for fractional calculus is extensive, and can be used to easily solve many problems of fractional calculus and fractional differential equations. In the future, we will make use of this theory to expand our research field to applied mathematics and fractional calculus.

REFERENCES

[1] M. Mirzazadeh, M. Ekici, A. Sonmezoglu, S. Ortakaya, M. Eslami, and A. Biswas, Soliton solutions to a few fractional nonlinear evolution equations in shallow water wave dynamics, European Physical Journal Plus, 2016, 131(5), 166.

[2] T. Das, U. Ghosh, S. Sarkar, and S. Das, Time independent fractional Schrödinger equation for generalized Mie-type potential in higher dimension framed with Jumarie type fractional derivative, Journal of Mathematical Physics, 2018, 59(2), 022111.

[3] S. Biswas, U. Ghosh, Approximate solution of homogeneous and nonhomogeneous $5\alpha^{th}$ order space-time fractional KdV, International Journal of Computational Methods, 2021, 18(1), 2050018.

[4] G. Shimin, M. Liquan, Z. Zhengqiang, J. Yutao, Finite difference/spectral-Galerkin method for a two-dimensional distributed-order time-space fractional reaction-diffusion equation, Applied Mathematics Letters, 2018, 85, pp. 157-163.

[5] Mohd. Farman Ali, Manoj Sharma, Renu Jain, An application of fractional calculus in electrical engineering, Advanced Engineering Technology and Application, 2016, 5(2), pp, 41-45.

[6] F. Mainardi, Fractional Calculus: Theory and Applications, Mathematics 2018, 6(9), 145.

[7] V. E. Tarasov, Mathematical economics: application of fractional calculus, Mathematics, 2020, 8(5), 660.

[8] Hasan A. Fallahgoul, Sergio M. Focardi and Frank J. Fabozzi, Fractional Calculus and Fractional Processes with Applications to Financial Economics, Academic Press, 2017.

[9] B. Carmichael, H. Babahosseini, SN Mahmoodi, M. Agah, The fractional viscoelastic response of human breast tissue cells, Physical Biology, 2015,12(4), 046001.

[10] J. T. Machado, Fractional Calculus: Application in Modeling and Control, Springer New York, 2013.

[11] Teodor M. Atanacković, Stevan Pilipović, Bogoljub Stanković, Dušan Zorica, Fractional Calculus with Applications in Mechanics: Vibrations and Diffusion Processes, John Wiley & Sons, Inc., 2014.

[12] N. Heymans, Dynamic measurements in long-memory materials: fractional calculus evaluation of approach to steady state, Journal of Vibration and Control, 2008, 14(9), pp.1587-1596.

[13] I. Petráš, Fractional-Order Nonlinear Systems, Springer, Berlin, 2011.

[14] L. Debnath, Recent applications of fractional calculus to science and engineering, International Journal of Mathematics and Mathematical Sciences, 2003, 54, pp. 3413-3442.

[15] M. Benchohra, S. Bouriah, J. J. Nieto, Existence and Ulam stability for nonlinear implicit differential equations with Riemann-Liouville fractional derivative, Demonstratio Mathematica, 2019, 52(1), pp. 437-450.

[16] W. Xu, W. Xu, S. Zhang, The averaging principle for stochastic differential equations with Caputo fractional derivative, Applied Mathematics Letters, 2019, 93, pp. 79-84.

[17] F. Ma , D. Jin , H. Yao, Theory analysis of Grunwald-Letnikov fractional derivative, Natural Sciences Journal of Harbin Normal University, 2011, 27(3), pp. 32-34.

[18] S. Kong, Y. Liu, New solution of Jumarie's modified R-L fractional equation, Computer Engineering and Applications, 2017, 53(5), pp. 28-30.

[19] G. Jumarie, Modified Riemann-Liouville derivative and fractional Taylor series of nondifferentiable functions further results, 2006, Computers and Mathematics with Applications, 2006, 51, pp.1367-1376.

[20] U. Ghosh, S. Sengupta, S. Sarkar, and S. Das, Analytic solution of linear fractional differential equation with Jumarie derivative in term of Mittag-Leffler function, American Journal of Mathematical Analysis, 2015, 3(2), pp.32-38.

[21] J. C. Prajapati, Certain properties of Mittag-Leffler function with argument x^{α}, $\alpha > 0$, Italian Journal of Pure and Applied Mathematics, 2013, 30, pp. 411-416.

[22] C. -H. Yu, Differential properties of fractional functions, International Journal of Novel Research in Interdisciplinary Studies, 2020, 7(5), pp.1-14.

[23] C. -H. Yu, Fractional Clairaut's differential equation and its application, International Journal of Computer Science and Information Technology Research, 2020, 8(4), pp. 46-49.

[24] C. -H. Yu, Separable fractional differential equations, International Journal of Mathematics and Physical Sciences Research, 2020, 8(2), pp. 30-34.

[25] C. -H. Yu, Integral form of particular solution of nonhomogeneous linear fractional differential equation with constant coefficients, International Journal of Novel Research in Engineering and Science, 2020, 7(2), pp. 1-9.

[26] C. -H. Yu, A study of exact fractional differential equations, International Journal of Interdisciplinary Research and Innovations, 2020, 8(4), pp. 100-105.

[27] C. -H. Yu, Research on first order linear fractional differential equations, International Journal of Engineering Research and Reviews, 2020, 8(4), pp. 33-37.

[28] C. -H. Yu, Method for solving fractional Bernoulli's differential equation, International Journal of Science and Research, 2020, 9(11), pp. 1684-1686.

[29] C. -H. Yu, Using integrating factor method to solve some types of fractional differential equations, World Journal of Innovative Research, 2020, 9(5), pp. 161-164.